Integrable Particle Dynamics in Accelerators

Lecture 2: Integration of Equations of Motion

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Equations of Motion

- All of these are supposed to give the same results
- Newton's second law:

$$\frac{d^2\vec{r}}{dt^2} = -\vec{\nabla}\Phi(\vec{r})$$

  - Complicated vector arithmetic & coordinate system dependence
- Lagrangian Formalism:

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0$$

  - n second-order differential equations
- Hamiltonian Formalism:

$$\frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i \qquad \frac{\partial \mathcal{H}}{\partial q_i} = -\dot{p}_i$$

  - $2n$ first-order differential equations
- Hamilton-Jacobi equation:

$$\mathcal{H}\left(\frac{\partial S}{\partial q_i}, q_i\right) = E$$

  - $S(\vec{q}, \vec{p})$ is a generator of canonical transformation $(\vec{q}, \vec{p}) \rightarrow \left(\vec{Q}, \vec{P}\right)$ for which $H(\vec{q}, \vec{p}) \rightarrow H'(\vec{P})$. If $S(\vec{q}, \vec{p})$ is separable, then the Hamilton-Jacobi equation breaks up into n ordinary differential equations which can be solved by simple quadrature. The resulting equations of motion are:

$$P_i(t) = P_i(0); \quad Q_i(t) = \frac{\partial H'}{\partial P_i}t + k_i$$

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Constants of motion

Constants of Motion: any function $C(\vec{q}, \vec{p}, t)$ of the generalized coordinates, conjugate momenta and time that is constant along **every** orbit, i.e., if $\vec{q}(t)$ and $\vec{p}(t)$ are a solution to the equations of motion, then

$$C[\vec{q}(t_1), \vec{p}(t_1), t_1] = C[\vec{q}(t_2), \vec{p}(t_2), t_2]$$

for any t_1 and t_2. The value of the constant of motion depends on the orbit, but different orbits may have the same numerical value of C

A dynamical system with n **degrees of freedom** always has $2n$ independent **constants of motion.** Let $q_i = q_i[\vec{q}_0, \vec{p}_0, t]$ and $p_i = p_i[\vec{q}_0, \vec{p}_0, t]$ describe the solutions to the equations of motion. In principle, these can be inverted to $2n$ relations $q_{i,0} = q_{i,0}[\vec{q}(t), \vec{p}(t), t]$ and $p_{i,0} = p_{i,0}[\vec{q}(t), \vec{p}(t), t]$. By their very construction, these are $2n$ **constants of motion.**

If $\Phi(\vec{x}, t) = \Phi(\vec{x})$, one of these $2n$ relations can be used to eliminate t. This leaves $2n - 1$ non-trivial constants of motion, which restricts the system to a $2n - (2n - 1) = 1$-dimensional surface in **phase-space,** namely the **phase-space trajectory** $\Gamma(t)$

Note that the elimination of time reflects the fact that the physics are **invariant** to time translations $t \rightarrow t + t_0$, i.e., the time at which we pick our initial conditions can not hold any information regarding our dynamical system.

Integrals of Motion I

Integrals of Motion: any function $I(\vec{x}, \vec{v})$ of the phase-space coordinates $(\vec{x}, \vec{v})$ **alone** that is constant along **every orbit**, i.e.

$$I[\vec{x}(t_1), \vec{v}(t_1)] = I[\vec{x}(t_2), \vec{v}(t_2)]$$

for any t_1 and t_2. The value of the integral of motion can be the same for different orbits. Note that an integral of motion can not depend on time. Thus, all integrals are constants, but not all constants are integrals.

Integrals of motion come in two kinds:

Isolating Integrals of Motion: these reduce the **dimensionality** of the trajectory $\Gamma(t)$ by one. Therefore, a trajectory in a dynamical system with n degrees of freedom and with i **isolating integrals of motion** is restricted to a $2n - i$ dimensional **manifold** in the $2n$-dimensional **phase-space**. Isolating integrals of motion are of great practical and theoretical importance.

Non-Isolating Integrals of Motion: these are integrals of motion that do **not** reduce the dimensionality of $\Gamma(t)$. They are of essentially no practical value for the dynamics of the system.

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Quadrature

- Integration by quadrature either means solving an integral analytically (i.e., symbolically in terms of known functions), or solving of an integral numerically (e.g., Gaussian quadrature, Newton-Cotes formulas).

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Integrating by Quadrature I

- Let's consider a 1D (*n* = 1) conservative Hamiltonian system

$$H(q, p) = \frac{p^2}{2} + V(q)$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -\frac{dV}{dq}$$

$$\dot{q} = \frac{\partial H}{\partial p} = p$$

- The equations of motion are

$$\ddot{q} = -\frac{dV(q)}{dq}$$

$$\dot{q}\ddot{q} + \dot{q}\frac{dV(q)}{dq} = 0$$

$$\frac{d}{dt}\left(\frac{\dot{q}^2}{2} + V(q) \right) = 0 \quad \text{, therefore} \quad \frac{\dot{q}^2}{2} + V(q) = I_1 = \text{const}$$

- The constant (in time) function I_1 is an integral of motion (1st isolating integral)

$$\frac{dq}{dt} = \sqrt{2(I_1 - V(q))} \qquad t + I_2 = \int \frac{dq}{\sqrt{2(I_1 - V(q))}}$$

-- now integrate by quadrature and invert to obtain *q(t)*

- I_2 is a trivial non-isolating integral

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Integrating by Quadrature II

- Notice that there have been four steps in this procedure

1. Identification of the first integral, I_1.
2. Use of the integral I_1 to reduce the order of the differential equation by one.
3. An explicit "integration by quadrature".
   - Beyond some simple polynomial potentials, this can be done only numerically
4. An inversion to obtain a single-valued solution $q(t)$
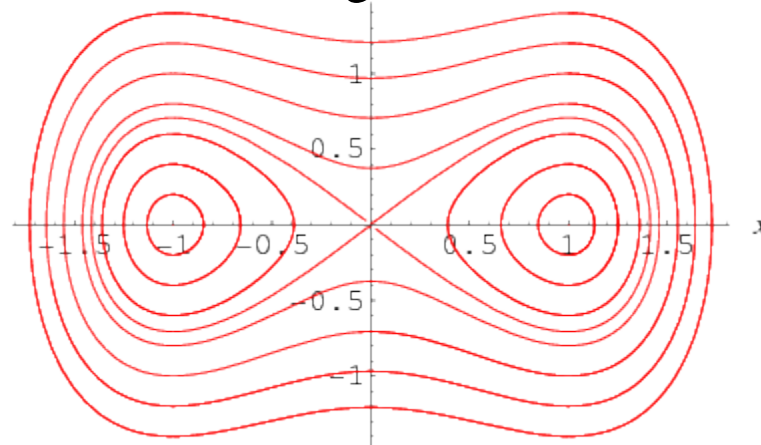   - May be very complicated

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Dynamics in Phase Space

simple harmonic oscillator
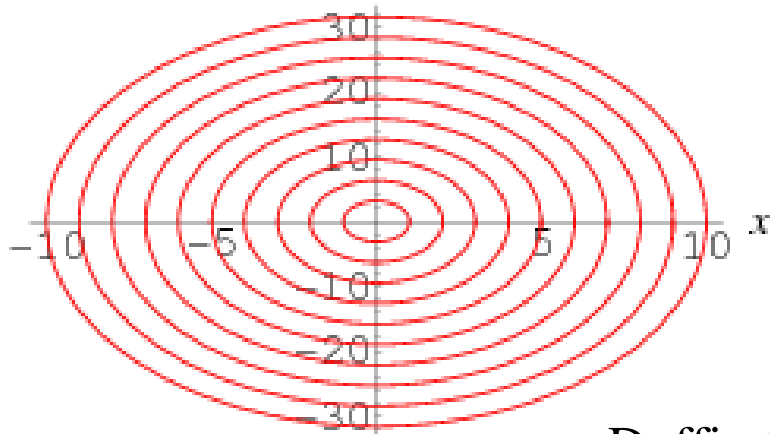
pendulum
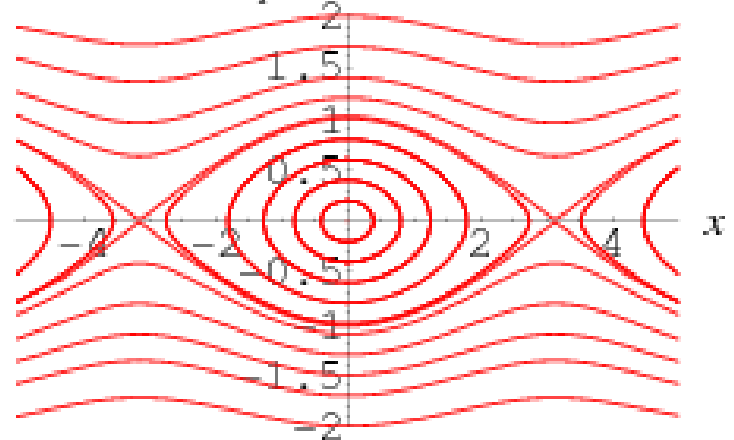
Duffing oscillator
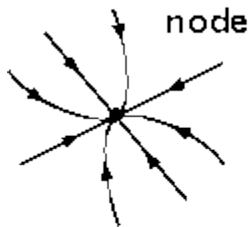
$$T = 2 \int_{x\min}^{x\max} \frac{dq}{\sqrt{2(E - V(q))}} = \frac{2\pi}{\omega(E)}$$ -- period of oscillations (a function of energy)
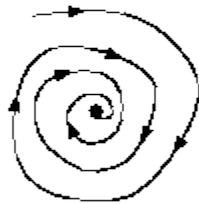
Taxonomy of 2D fixed points

- The fixed points are those values of p_0 and q_0 for which the phase flow is stationary:

$$\dot{p} = 0; \ \dot{q} = 0 \text{ at } p = p_0 \text{ and } q = q_0$$
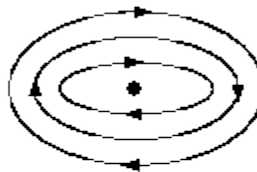
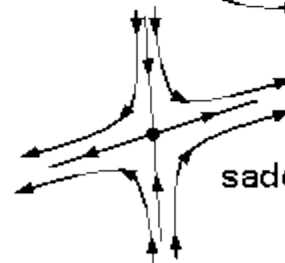Stable (attractors) — node, spiral

Neutral — center

Unstable (repellers) — node, spiral, saddle

Time-dependent systems

- Integrals of motion for time-dependent (non-autonomous) systems (such as the Courant-Snyder invariant) are extremely rare and may require some luck to discover.

- In this class we will learn about two classes of time-dependent periodic systems, applicable to accelerators:

  ➤ Systems, where time-dependence is eliminated by transforming the time variable (similar to the Courant-Snyder invariant)

  ➤ And systems, where time dependence is manifested in special delta-function like "kicks". Such systems will be called "integrable mappings".

Beyond $n = 1$

- For dynamical systems with $n = 1$, we can integrate the pair of first-order diff. equations.

- At the end of 19th century all dynamical systems (for $n > 1$) were thought to be integrable.
  - ➢ 1885 math. Prize was established for finding the solution of an n-body problem (n>2)

- However, nonintegrable systems constitute the majority of all real-world systems (1st example, H. Poincare, 1895)
  - ➢ The phase space of a simple 3-body system is far from simple. This plot of velocity versus position is called a homoclinic tangle.

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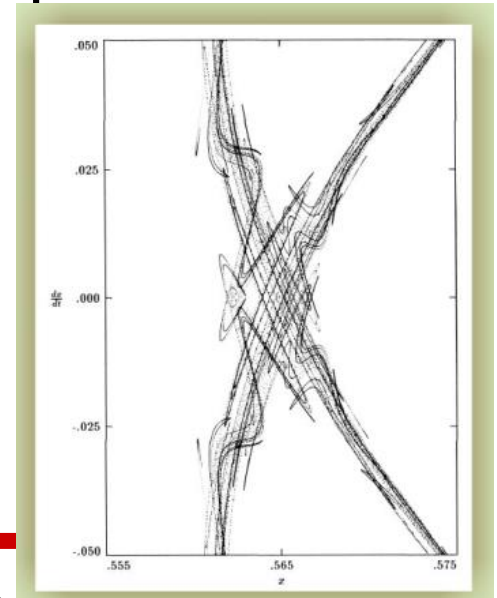
Symplectic matrix

- The symmetry of Hamilton's equations allows to consider the variables p_i and q_i on an equal footing.

  If $\qquad \mathbf{z} = (q_1, ..., q_n, p_1, ... p_n)$

- Hamilton's equations can be written as:

$$\dot{\mathbf{z}} = \mathbf{J} \cdot \nabla H(\mathbf{z})$$

where the matrix $\mathbf{J}$ is the 2nx2n *symplectic* matrix

$$\mathbf{J} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix}$$

and $\mathbf{I}$ is the nxn unit matrix.

Integrals of Motion II

- A conservative (autonomous) Hamiltonitan system (i.e. H(q,p,t) = H(q,p)) with n degrees of freedom may have between 1 and $2n - 1$ isolating integrals of motion.

- Definition: Two functions $I_1(\vec{q}, \vec{p})$ and $I_2(\vec{q}, \vec{p})$ are said to be in involution if their Poisson bracket vanishes, i.e. if

$$[I_1, I_2] = \sum_{i=1}^{n} \frac{\partial I_1}{\partial q_i} \frac{\partial I_2}{\partial p_i} - \frac{\partial I_1}{\partial p_i} \frac{\partial I_2}{\partial q_i} = 0$$

Poisson Brackets

$$[f,g] = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}$$

$$[f,g] = -[g,f],$$

$$[f,c] = 0 \text{ for } c \text{ a constant,}$$

$$[f_1 + f_2, g] = [f_1,g] + [f_2,g],$$

$$[f_1 f_2, g] = f_1[f_2,g] + [f_1,g]f_2,$$

$$\frac{\partial}{\partial t}[f,g] = \left[\frac{\partial f}{\partial t}, g\right] + \left[f, \frac{\partial g}{\partial t}\right].$$

$$\{q_i, q_j\} = 0$$

$$\{p_i, p_j\} = 0$$

$$\{q_i, p_j\} = \delta_{ij}$$

$$[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} f(p, q, t) = \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial f}{\partial t}$$

$$= \{f, H\} + \frac{\partial f}{\partial t}.$$

If a quantity *f(p,q)* is explicitly time independent and it is in involution with *H* , then *f(p,q)* is an integral of motion.

Integrals of Motion III

- It is obvious that for an autonomous system H is an integral of motion, $[H, H] = 0$. And so is any function of H : $[H, f(H)] = 0$

- Isolating integrals must be functionally independent of each other!

Integrals of Motion IV

- How many integrals of motion does one need to "solve" the dynamical equations of motion?

- In general, a system of n first-order diff. equations requires $n-1$ constants (integrals) in order to effect a complete "integration".

- The Hamiltonian system has $2n$ equations. Does it mean we require $2n-1$ integrals?

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How many?

- The Hamiltonian system has *2n* equations. Does it mean we require *2n-1* integrals?

- Answer: It turns out, because of symmetric nature of Hamilton's equation (a.k.a. the symplectic nature), we need only *n* integrals of motion.
  - ➤ This miracle occurs due to canonical transformations

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Liouville-Arnold Theorem

- The Liouville–Arnold theorem states that if, in a Hamiltonian dynamical system with n degrees of freedom, there are also known n first integrals of motion that are independent and in involution, then there exists a canonical transformation to action-angle coordinates in which the transformed Hamiltonian is dependent only upon the action coordinates and the angle coordinates evolve linearly in time. Thus the equations of motion for the system can be solved in quadratures if the canonical transform is explicitly known.

Canonical Transformations

- Goal: To find transformations

$$Q_i = Q_i(q_1, \ldots, q_n, p_1, \ldots, p_n, t) \quad P_i = P_i(q_1, \ldots, q_n, p_1, \ldots, p_n, t)$$

that satisfy Hamilton's equation of motion

$$\dot{q}_i = \frac{\partial H}{dp_i} \quad \dot{p}_i = -\frac{\partial H}{dq_i} \quad \Longrightarrow \quad \dot{Q}_i = \frac{\partial K}{dP_i} \quad \dot{P}_i = -\frac{\partial K}{dQ_i}$$

  - K is the transformed Hamiltonian $K = K(Q, P, t)$

- Hamilton's principle requires

$$\delta \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H(q, p, t) \right) dt = 0 \quad \text{and} \quad \delta \int_{t_1}^{t_2} \left(P_i \dot{Q}_i - K(Q, P, t) \right) dt = 0$$

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Phase-space volume is preserved under canonical transformations

$$\int \prod_{i=1}^{n} dP_i dQ_i = \int \prod_{i=1}^{n} dp_i dq_i \qquad \text{-- one of Poincare invariants}$$

- Therefore, the canonical transformation must have a unit Jacobian. Which of these could be canonical transformations?

$$Q = -p; \quad P = q$$

$$q = P \cos Q; \quad p = P \sin Q$$

$$q = \sqrt{P} \sin Q; \quad p = \sqrt{P} \cos Q$$

$$Q = 2q; \quad P = \frac{p}{2}$$

$$Q = 2q; \quad P = 2p$$

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Canonical transformation

$$P_i \dot{Q}_i - K + \frac{dF}{dt} = p_i \dot{q}_i - H$$

- Hamilton's principle

$$\delta \int_{t_1}^{t_2} \left(P_i \dot{Q}_i - K \right) dt = \delta \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H - \frac{dF}{dt} \right) dt = -\delta \left[F \right]_{t_1}^{t_2} = 0$$

- Satisfied if $\delta p = \delta q = \delta P = \delta Q = 0$ at t_1 and t_2

- F can be any function of p_i, q_i, P_i, Q_i and t

 - It defines a canonical transformation
 - Call it the generating function of the transformation

or generator

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Type-1 Generator

- $F = F(q, Q)$ is not very general
    - It does not allow t-dependent transformation
    - Fix this by extending to $F = F_1(q, Q, t)$ ⇐ Call it Type-1

$$p_i = \frac{\partial F_1(q, Q, t)}{\partial q_i} \qquad P_i = -\frac{\partial F_1(q, Q, t)}{\partial Q_i}$$

    - This affects the Hamiltonian

$$\frac{dF}{dt} = \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i + \frac{\partial F_1}{\partial t} = p_i\dot{q}_i - P_i\dot{Q}_i + K - H$$

⟹ $K = H + \dfrac{\partial F_1}{\partial t}$

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Harmonic Oscillator

- Consider a 1-dimensional harmonic oscillator

$$H(q, p) = \frac{p^2}{2m} + \frac{kq^2}{2} = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) \qquad \omega^2 \equiv \frac{k}{m}$$

  - Sum of squares → Can we make them sine and cosine?

  - Suppose $p = f(P)\cos Q$ $\qquad q = \dfrac{f(P)}{m\omega}\sin Q$

  ➡ $K = H = \dfrac{\{f(P)\}^2}{2m}$ ⬅ Q is cyclic → P is constant

  - Trick is to find $f(P)$ so that the transformation is canonical
    - How?

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Harmonic Oscillator

- Let's try a Type-1 generator

$$F_1(q,Q,t) \qquad p = \frac{\partial F_1}{\partial q} \qquad P = -\frac{\partial F_1}{\partial Q}$$

  - Express p as a function of q and Q

$$p = f(P)\cos Q \qquad q = \frac{f(P)}{m\omega}\sin Q \implies p = m\omega q \cot Q$$

  - Integrate with q $\implies$ $F_1 = \frac{m\omega q^2}{2}\cot Q$

$$\implies P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2 Q}$$

We are getting somewhere

Harmonic Oscillator

$$p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q \qquad P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2 Q}$$

- We need to turn $H(q, p)$ into $K(Q, P)$
- Solve the above equations for q and p

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q \qquad p = \sqrt{2Pm\omega} \cos Q$$

- Now work out the Hamiltonian

$$K = H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) = \omega P$$

- Things don't get much simpler than this…

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Harmonic Oscillator

$$K = \omega P = E$$

- Solving the problem is trivial

$$P = \text{const} = \frac{E}{\omega} \qquad \dot{Q} = \frac{\partial K}{\partial P} = \omega \qquad Q = \omega t + \alpha$$
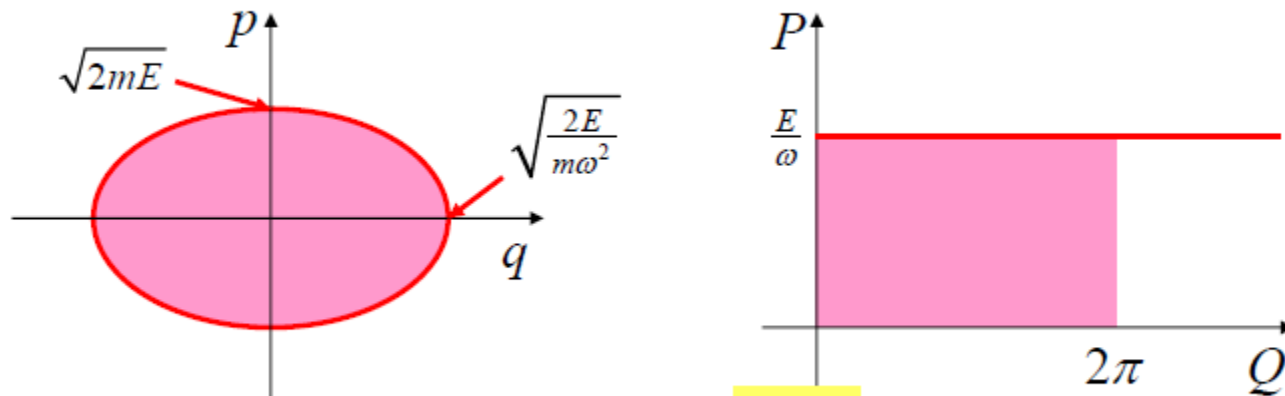
Finally

$$p = \sqrt{2Pm\omega}\cos Q = \sqrt{2mE}\cos(\omega t + \alpha)$$

$$q = \sqrt{\frac{2P}{m\omega}}\sin Q = \sqrt{\frac{2E}{m\omega^2}}\sin(\omega t + \alpha)$$

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- Oscillator moves in the p-q and P-Q phase spaces

$\sqrt{2mE}$

$\sqrt{\dfrac{2E}{m\omega^2}}$

$\dfrac{E}{\omega}$

2π

- One cycle draws the same area $\dfrac{2\pi E}{\omega}$ in both spaces

- Notice that the phase space volume is preserved

Four Basic Generators

Generator	Derivatives	Trivial Case
$F_1(q,Q,t)$	$p_i = \dfrac{\partial F_1}{\partial q_i} \qquad P_i = -\dfrac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i \qquad \begin{array}{l} Q_i = p_i \\ P_i = -q_i \end{array}$
$F_2(q,P,t) - Q_i P_i$	$p_i = \dfrac{\partial F_2}{\partial q_i} \qquad Q_i = \dfrac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i \qquad \begin{array}{l} Q_i = q_i \\ P_i = p_i \end{array}$
$F_3(p,Q,t) + q_i p_i$	$q_i = -\dfrac{\partial F_3}{\partial p_i} \quad P_i = -\dfrac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i \qquad \begin{array}{l} Q_i = -q_i \\ P_i = -p_i \end{array}$
$F_4(p,P,t) + q_i p_i - Q_i P_i$	$q_i = -\dfrac{\partial F_4}{\partial p_i} \quad Q_i = \dfrac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i \qquad \begin{array}{l} Q_i = p_i \\ P_i = -q_i \end{array}$

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Summary (I)

- ## Canonical transformations

  $$P_i \dot{Q}_i - K + \frac{dF}{dt} = p_i \dot{q}_i - H$$

  ➢ Hamiltonian formalism is invariant under canonical transformations

  ➢ Preserve phase-space volume

  ➢ Generating functions define canonical transformations

- ## What does it have to do with integrable dynamical systems?

The Optimal Transformation I

- The practical use of canonical transformations is to find those that make the integration of Hamilton's equations as simple as possible. The optimal case is when the transformed Hamiltonian depends only on the new momenta, P_i (like in our Harmonic oscill example)

$$H(p_1,...,p_n,q_1,...,q_n) \rightarrow K(P_1,...,P_n)$$

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0, \quad \text{i.e., } P_i = const$$

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = f_i\left(P_i,...,P_n\right)$$

- The equations for Q's can be immediately integrated

$$Q_i = f_i t + \delta_i$$

- The *n* momenta P_i are the integrals of motion, that enable us to perform the integration.

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The optimal transformation II

- We can now transform the solution to our "old" original p_i and q_i.

- Of course, we have to be able to do two things:
  1. Find these magical new variables P
  2. And, know how to correctly transform the Hamiltonian into its new representation.

The optimal transformation III

LEMMA: If a system with n degrees of freedom has n constants of motion $P_i(\vec{q}, \vec{p}, t)$ [or integrals of motion $P_i(\vec{q}, \vec{p})$] that are in involution, then there will also be a set of n functions $Q_i(\vec{q}, \vec{p}, t)$ [or $Q_i(\vec{q}, \vec{p})$] which together with the P_i constitute a set of canonical variables.

Thus, given n isolating integrals of motion $I_i(\vec{q}, \vec{p})$ we can make a canonical transformation $(\vec{q}, \vec{p}) \rightarrow (\vec{Q}, \vec{P})$ with $P_i = I_i(\vec{q}, \vec{p}) =$ constant and with $Q_i(t) = \Omega_i t + k_i$

An integrable, Hamiltonian system with n degrees of freedom always has a set of n isolating integrals of motion in involution. Consequently, the trajectory $\Gamma(t)$ is confined to a $2n - n = n$-dimensional manifold phase-space.

The surfaces specified by $(I_1, I_2, .., I_n) =$ constant are topologically equivalent to n-dimensional tori. These are called invariant tori, because any orbit originating on one of them remains there indefinitely.

In an integrable, Hamiltonian system phase-space is completely filled (one says 'foliated') with invariant tori.

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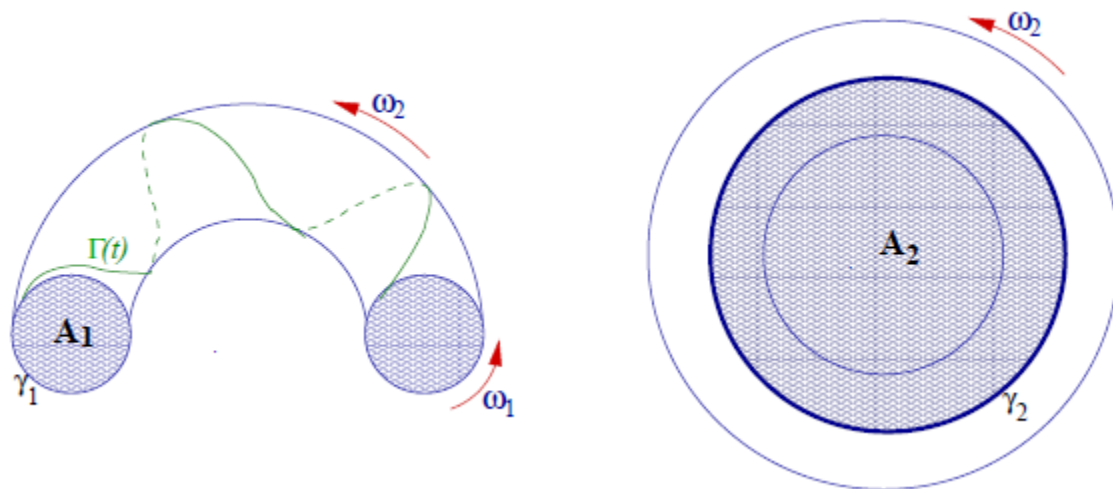
Motion on the tori
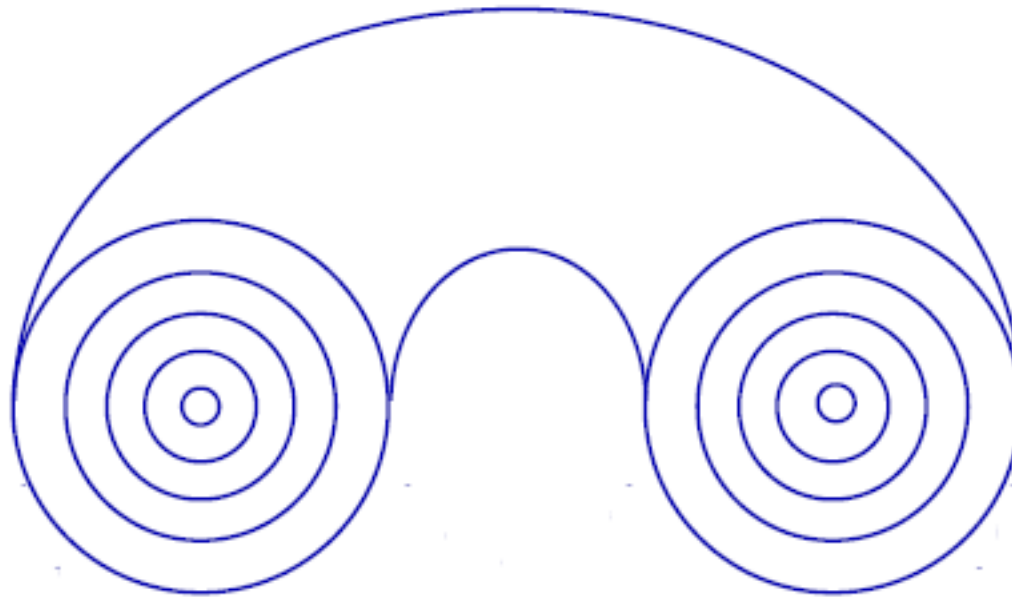

The **action variables** are defined by:

$$J_i = \frac{1}{2\pi} \oint_{\gamma_i} \vec{p} \cdot \mathrm{d}\vec{q}$$

with γ_i the closed loop that bounds cross section A_i.

Motion on the tori

In an integrable, Hamiltonian system phase-space is completely foliated with non-intersecting, invariant tori

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One might think at this point, that one has to take $P_i = I_i$. However, this choice is **not unique**. Consider an integrable Hamiltonian with $n = 2$ degrees of freedom and let I_1 and I_2 be two isolating integrals of motion in involution. Now define $I_a = \frac{1}{2}(I_1 + I_2)$ and $I_b = \frac{1}{2}(I_1 - I_2)$, then it is straightforward to proof that $[I_a, I_b] = 0$, and thus that (I_a, I_b) is also a set of isolating integrals of motion in involution. In fact, one can construct infinitely many sets of isolating integral of motion in involution. Which one should we choose, and in particular, which one yields the most meaningful description of the **invariant tori**?

Answer: The Action-Angle variables

- The idea of action-angle variables is to find the pair of conjugate variables such that the conjugate "coordinate" increases by 2π after each complete period of motion .

Hamilton-Jacobi Equation

- Our goal is to find the canonical transformation to a set of constant conjugate momenta. We will use the F_2 type generator,

$$S = S(q_i, \alpha_i)$$

$$p_i = \frac{\partial S}{\partial q_i}; \quad \beta_i = \frac{\partial S}{\partial \alpha_i}$$

where the β_i are the "new" coordinates. We obtain the Hamilton-Jacobi equation for S in n independent variables

$$H\left(q_i, \frac{\partial S}{\partial q_i} \right) = K(\alpha_i)$$

Here, the right-hand side is to be viewed as a constant quantity.

Solving this equation is just as difficult as the canonical equations of motion, except for several classes of dynamical systems: (1) one-degree of freedom and (2) separable

Action-Angle variables in 1D

$$H\left(q, \frac{\partial S}{\partial q} \right) = E = K(I)$$

$$p = \frac{\partial S}{\partial q}; \quad \theta = \frac{\partial S}{\partial I}$$

$$\frac{d\theta}{dq} = \frac{\partial}{\partial I}\left(\frac{dS}{dq} \right)$$

$$2\pi = \oint d\theta = \frac{\partial}{\partial I}\oint \frac{\partial S}{\partial q} dq = \frac{\partial}{\partial I}\oint p\, dq$$

$$I(E) = \frac{1}{2\pi}\oint p(q, E)dq$$

- This is the definition of the action variable
- In 1D, its value does not depend on the choice of p and q

Action-Angle variables in 1D

- We can now invert $I(E) \rightarrow E(I)=K(I)$
- The canonical equations of motion can now be solved:

$$I = const; \quad \theta = \frac{dK}{dI}t + \delta = \omega(I)t + \delta$$

Separable systems (n-degrees of freedom)

- The H-J equation for $n > 1$ cannot, in general, be solved unless it is separable, i.e.

$$S = \sum_{i=1}^{n} S_i(q_i, \alpha_1, ..., \alpha_n)$$

- Then $p_i = \dfrac{\partial S}{\partial q_i}$ is a function of only one coordinate

and we can define a set of action variables

$$I_i = \frac{1}{2\pi} \oint p_i(q_i, \alpha_1, ..., \alpha_n) dq_i$$

Separable systems

- A rather simple example: $H = \sum_{i=1}^{n} H_i(q_i, p_i)$

- More examples of separable systems can be found Landau and Lifshitz "Mechanics"

- Some systems are separable in multiple coordinate systems (e.g. Cartesian, polar, …)
  - ➢ Then, the Action-Angle variables are not unique.

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Summary

- We are trying to draw a distinction between integrable and non-integrable systems. The latter can exhibit chaotic behavior (leading to particle losses in accelerators), whereas the former exhibits stable periodic behavior.

- The question remains: given a system of equations, how can one tell a priori whether or not they are integrable?

- We will present several accelerator focusing systems, where we start the design with a non-linear integrable system.

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Summary

- The definition of integrability is simple to state: an autonomous n-degree of freedom Hamiltonian is integrable if N independent integrals of motion exist and these are in involution with each other. However, a failure to find such a set of global invariants does not exclude the possibility that the Hamiltonian system in question is integrable.

Extra slides

Simple example 1

$$P_i \dot{Q}_i - K + \frac{dF}{dt} = p_i \dot{q}_i - H$$

- Try a generating function: $F = q_i P_i - Q_i P_i$
  - Canonical transformation generated by F is

$$P_i \dot{Q}_i - K + \frac{dF}{dt} = -K + (q_i - Q_i)\dot{P}_i + P_i \dot{q}_i = p_i \dot{q}_i - H$$

➡️ $Q_i = q_i$ $P_i = p_i$ ⬅️ Identity transformation

$K = H$

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- Let's try this one: $F = f_i(q_1, \ldots, q_n, t)P_i - Q_i P_i$
  - f_i are arbitrary functions of $q_1 \ldots q_n$ and t

$$P_i \dot{Q}_i - K + \frac{dF}{dt} = -K + (f_i - Q_i)\dot{P}_i + P_i \frac{\partial f_i}{\partial q_j}\dot{q}_j + \frac{\partial f_i}{\partial t}P_i = p_i \dot{q}_i - H$$

➡ $Q_i = f_i(q_1, \ldots, q_n, t)$ ◄— All "point transformations" of generalized coordinates are covered

$$p_i = \frac{\partial f_j}{\partial q_i}P_j$$ ◄— Must invert these n equations to get P_i

$$K = H + \frac{\partial f_i}{\partial t}P_i$$

- Let's try one example:

$$F = F_2(q, P) - QP = q^3 P - QP \text{ -- generator}$$

Finding the Generator

- Let's look for a generating function
  - Suppose $K(Q, P, t) = H(q, p, t)$ for simplicity

  $$\Rightarrow \quad \frac{dF}{dt} = p_i \dot{q}_i - P_i \dot{Q}_i$$

  - Easiest way to satisfy this would be

  $$F = F(q, Q) \qquad \frac{\partial F}{\partial q_i} = p_i \qquad \frac{\partial F}{\partial Q_i} = -P_i$$

- Trivial example: $F(q, Q) = q_i Q_i$

  $$\Rightarrow \quad p_i = Q_i \qquad P_i = -q_i$$

  In the Hamiltonian formalism, you can freely swap the coordinates and the momenta

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